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One-loop derivation of the Wilson polygon–MHV amplitude duality

A Gorsky[1,4] and A Zhiboedov[2,3]

[1] Institute of Theoretical and Experimental Physics, Moscow, Russia
[2] Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, Dubna, Russia
[3] Physics Department, Moscow State University, Moscow, Russia
[4] FTPI, University of Minnesota, Minneapolis, MN 55455, USA

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Abstract

We discuss the origin of the Wilson polygon–MHV amplitude duality at a perturbative level. It is shown that the duality for the MHV amplitudes at the one-loop level can be proven upon a particular change of variables in Feynman parametrization and with the use of the relation between Feynman integrals at different space-time dimensions. Some generalization of the duality which implies the insertion of a particular vertex operator at the Wilson triangle is found for the 3-point function. We discuss the analytical structure of Wilson loop diagrams and present the corresponding Landau equations. The geometrical interpretation of the loop diagram in terms of the hyperbolic geometry is discussed.

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1. Introduction

The clarification of the geometrical structure behind perturbation theory in SYM which would provide a method of summation of the series remains a challenging problem. In recent years two novel ideas concerning these issues have been developed. It was demonstrated in [1] that important localization phenomena occur for the perturbative amplitude in twistor space. On the other hand the stringy calculation of the amplitudes [2] suggested a hidden duality between the amplitudes in $\mathcal{N} = 4$ SYM and the Wilson polygon built from light-like momenta of external gluons. It is important to note that the amplitudes look to be mapped to an ordinary position space Wilson loop. A connection between amplitudes and momentum space Wilson loops was investigated in [3].

This duality has been checked at one- [4, 5] and two loops [6, 7] in the perturbative theory and has the possibility to be all-loop exact (see [11] for a review). During this development it was also realized that there is an underlying important dual superconformal symmetry which was clarified both in the weak coupling [8, 9] and strong coupling [10] cases. The dual superconformal symmetry was argued to be the consequence of the fermionic T-duality in the stringy sigma model [12] and the combination of the usual superconformal and dual superconformal symmetries implies Yangian symmetry in the perturbative $\mathcal{N} = 4$ SYM theory [13].

In spite of the impressive progress, many key issues are still to be clarified. In this paper we shall focus on the origin of the Wilson polygon–MHV amplitude duality which will be analyzed at the one-loop level. We shall try to get the precise mapping between the one-loop diagram for the MHV amplitude and the one-loop correction to the Wilson polygon. It turns out that the proper change of variables in the Feynman parametrization of the loop integral for the six-dimensional box diagram brings it to the form of the Wilson polygon in four dimensions. Conversely, the four-dimensional box diagram can be related to the Wilson polygon is six dimensions. The IR divergences of the amplitudes become mapped into the UV divergences of the Wilson polygon. Moreover, it is seen that the MHV amplitude obeys this special property since it can be expressed in terms of the two-mass easy box diagrams only, and a simple change of variable we have found does not work for the non-MHV amplitudes. Using the known interplay between particular $D = 6$ and $D = 4$ integrals [14, 15], the answer can immediately be presented in terms of the finite part of the $D = 4$ two-mass easy box.

The loop amplitudes can be calculated via dispersion relations, and hence the duality implies that some version of the imaginary part calculations can be formulated for the loop corrections to the Wilson polygon as well. To this end we shall slightly generalize the cut technique for the loop diagrams and argue that on the Wilson polygon the dispersion calculation corresponds to cutting of the Wilson polygon into several pieces and subsequent gluing with the insertion of particular operators. We shall also comment on the Landau equations for the singularities on the Wilson polygon.

It is reasonable to search for a more natural geometry behind the one-loop calculation which would shed additional light on the duality under discussion. Let us first comment on the previous studies of this issue. The one-loop correlation functions suggest the natural emergence of an AdS-type geometry in 3-point [17] and 4-point functions [18]. A similar hyperbolic structure is also clearly seen in the one-loop effective action in the constant external field [19]. In both cases the Schwinger parametrization of the loop integral plays a crucial role. In particular for the 3-point function the combination of the Schwinger parameters plays the role of the radial coordinate in the AdS_5 [17], while in the effective action case a similar identification emerges in the AdS_3 submanifold [19].

The geometry behind the BDS formula [23] emerging upon summation over the loops was suggested in [24] and the corresponding fermionic representation which supports the hidden integrability was found. The key point is that there is a natural playground for the topological strings both in the A model with the Kähler gravity and B model involving KS gravity on the moduli space of the complex structures. Both complex and Kähler types of moduli are provided by the kinematical invariants of the scattering particles.

In this paper we shall discuss the geometrical aspect of the one-loop calculation based on the observation of [20] related to the Kähler moduli. It was found in [20] that the one-loop box integral counts the hyperbolic volume of the 3D manifold in the space of Feynman parameters. Contrary to Gopakumar's approach when the 4-point function is treated differently from the 3-point function, in this approach they are considered on an equal footing. Since the 3D hyperbolic manifolds emerge naturally as the knot complements we shall make some links with the Chern–Simons calculation with the inserted Wilson loop.

The paper is organized as follows. In section 2 we review the duality between the MHV amplitudes in $\mathcal{N} = 4$ theory and the Wilson polygon. In section 3 we briefly explain the relevant hyperbolic geometry behind the one-loop calculations. Section 4 is devoted to the explicit derivation of the duality for the MHV amplitude at the one-loop level. In section 5 we provide a simplified example of the duality for the 3-point function which involves the vertex operator on the Wilson polygon. In section 6 we consider some aspects of the unitarity calculation of the Wilson polygons. Section 7 is devoted to comments concerning the relation of the divergent contributions with the hyperbolic geometry of one-loop diagrams. In the last section we shall summarize our observations and mention some open problems.

2. The connection between Wilson polygons and MHV amplitudes

In this section we briefly review the conjectured duality between the loop amplitudes in $\mathcal{N} = 4$ theories and Wilson polygons built from the external momenta (see [11] for a review).

Specifically, it was conjectured in [2] that any MHV N-leg color-ordered amplitude follows from the vacuum expectation value of the Wilson loop in the special form

$$\frac{\mathcal{A}^{\text{MHV}}_{\text{all-loop}}}{\mathcal{A}^{\text{MHV}}_{\text{tree}}} = \langle W(x_1, x_2, \ldots, x_N)\rangle, \tag{1}$$

where the closed Wilson loop polygon has light-like momenta at the edges $p_i = x_i - x_{i+1}$ and vertices at x_i. Its closeness is provided by the total momentum conservation. At the strong coupling limit both Wilson polygon and the MHV amplitude are calculated in the sigma model approach.

At weak coupling, to check this Wilson polygon–MHV amplitude duality, one considers the expansion of the Wilson polygon in the YM coupling, treating the Wilson loop as placed in the coordinate space. In its weaker form the duality connects only the finite part of the two objects

$$\text{Fin}\left[\frac{\mathcal{A}^{\text{MHV}}_{\text{all-loop}}}{\mathcal{A}^{\text{MHV}}_{\text{tree}}}\right] = \text{Fin}[\langle W(p_1, p_2, \ldots, p_N)\rangle]. \tag{2}$$

Perfect matching of the Wilson loop and amplitude finite parts has been found for one- [4, 5] and two-loop [6, 7] solutions up to six external legs. Moreover, it was demonstrated that the anomalous Ward identities for the special conformal transformations of the form

$$\sum_{i=1}^{n} \left(2x_i^\nu x_i \partial_i - x_i^2 \partial_i^\nu\right) \text{Fin}[\log \mathcal{W}_n] = \frac{1}{2}\Gamma_{\text{cusp}} \sum_{i=1}^{n} \log \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\nu, \tag{3}$$

where Γ_{cusp} is the cusp anomalous dimension, fix the solution up to five external legs.

The anomalous Ward identities can be applied to both the amplitudes and the Wilson polygons; however, starting with six external legs the Ward identity allows an arbitrary function of the conformal cross-ratios, which cannot be fixed by the superconformal group arguments.

There are some specifics concerning the loop MHV amplitudes. The one-loop Wilson loop diagram with arbitrary number of external legs can be mapped to the finite part of the two-mass easy box, which is the main building block of the solution. The generalization of the duality to the non-MHV amplitudes turns out to be a nontrivial issue. In particular it is known [37] that the NMHV loop amplitude involves mass box diagrams as well and harder diagrams are relevant for the N^k MHV amplitudes. No recipe for the duality beyond the MHV case has yet been formulated.

It was demonstrated that the unitarity approach is fruitful for the description of the loop amplitudes. A general planar color-ordered one-loop scattering super-amplitude can be written in the following way:

$$\mathcal{A}_{n;1} = \mathrm{i}(2\pi)^4\delta^4(p)\sum(\mathcal{C}^{4m}I^{4m} + \mathcal{C}^{3m}I^{3m} + \mathcal{C}^{2mh}I^{2mh} + \mathcal{C}^{2me}I^{2me} + \mathcal{C}^{1m}I^{1m}), \tag{4}$$

where the I are the scalar-box integrals with the corresponding number of legs off-shell.

All one needs to calculate for a given amplitude are the coefficients, which can be done in terms of quadruple cuts. The general form of the $\mathcal{C}^m$ takes the form

$$\mathcal{C}^m = \delta^8\left(\sum_{i=1}^{n}\lambda_i\eta_i\right)\left[\mathcal{P}_{n;1}^{(0),m} + \mathcal{P}_{n;1}^{(4),m} + \cdots + \mathcal{P}_{n;1}^{(4n-16),m}\right], \tag{5}$$

where $\mathcal{P}_{n;1}^{(4k),m}$ are homogeneous polynomials of degree $4k$ in Grassmann variables.

The one-loop MHV super-amplitude takes the following form:

$$\mathcal{A}_{n;1}^{\mathrm{MHV}} = i(2\pi)^4\delta^4(p)\frac{\delta^8\left(\sum_{i=1}^{n}\lambda_i\eta_i\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\left[\sum_{s=3}^{n-1}I_{1,2,s,s+1}^{2me}\Delta_{1,2,s,s+1} + \text{cyclic}\right], \tag{6}$$

where $\Delta_{r,t,s,s+1} = -\frac{1}{2}\left[x_{sr}^2x_{s+1t}^2 - x_{s+1r}^2x_{st}^2\right]$. The solution is fully defined by two-mass easy boxes.

The general one-loop NMHV amplitude has a more complicated structure, namely

$$\mathcal{A}_{n;1}^{\mathrm{NMHV}} = \mathcal{A}_{n;1}^{\mathrm{MHV}}\left[\sum_{p,q,r=1}^{n}R_{pqr}\left(1 + \frac{\lambda}{8\pi^2}V_{pqr} + O(\varepsilon)\right)\right], \tag{7}$$

where two-mass hard and three-mass boxes are involved, R_{pqr} are dual superconformal and V_{pqr} are dual conformal invariants.

3. Hyperbolic geometry of one loop

In what follows it will be useful to utilize the geometrical picture behind the one-loop calculations which we shall review following [20]. Let us explain first the explicit map of the box diagram to the hyperbolic volume of the particular simplex build from the kinematical invariants of the external momenta. To this end, we introduce the Feynman parametrization of the internal generically massive propagators with the parameters α_i. If one considers the one-loop N-point function with the external momenta p_i in D space-time dimensions it can be brought into the usual form

$$J(D, p_1, \ldots p_N) \propto \int_0^1\prod_{i=1}^{N}\mathrm{d}\alpha_i\frac{\delta\left(1 - \sum_{i=1}^{N}\alpha_i\right)}{\left[\sum_{i=1}^{N}\alpha_i^2m_i^2 + \sum_{i<j}^{N}\alpha_i\alpha_jm_im_jC_{ij}\right]^{N-\frac{D}{2}}}, \tag{8}$$

where

$$C_{ij} = \frac{m_i^2 + m_j^2 - k_{ij}^2}{2m_im_j}, \qquad k_{ij} = \sum_{m=i}^{j-1}p_m \tag{9}$$

and m_i is the mass in the ith propagator.

It is possible [20] to organize for the generic one-loop diagram the N-dimensional *basic simplex* defined as follows. First introduce the N mass vectors m_ia_i, where a_i are the unit vectors. The length of the side connecting the ith and jth mass vectors is $\sqrt{k_{ij}^2}$—here we work

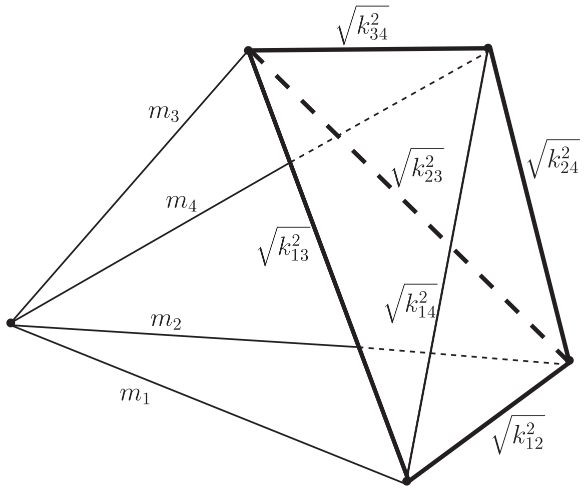

Figure 1. An example of the basic simplex for the box diagram.

in the kinematical region where $k_{ij}^2 > 0$—that is, one can define the momentum side of the simplex. For $N = 4$ see figure 1. The N-dimensional simplex involves $\frac{N(N+1)}{2}$ sides including N mass sides as well as $\frac{N(N-1)}{2}$ momentum sides. At each vertex N sides meet, and at all vertices but one there are one mass side and $(N-1)$ momentum sides. The volume of such an N-dimensional simplex is given as follows:

$$V^{(N)} = \frac{\left(\prod m_i\right)\sqrt{\det C}}{N!}, \tag{10}$$

where C is the matrix with elements C_{ij} defined above.

There are $(N+1)$ hypersurfaces of dimension $(N-1)$, one of which contains only momentum sides and can be related to the massless N-point function.

It is convenient to make a change of variables that transforms the loop integral into the following form:

$$J(D, p_1, \ldots p_N) \propto \int_0^\infty \prod_{i=1}^N \frac{\mathrm{d}\alpha_i}{m_i} \delta(\alpha^T C\alpha - 1) \left(\sum_{i=1}^N \frac{\alpha_i}{m_i}\right)^{N-D}, \tag{11}$$

that is, the integration is now over the quadrics in the space of the Feynman parameters.

For $D = N$ this integral is nothing but the content $\Omega^{(N)}$ of part of an $(N-1)$-dimensional non-Euclidian hypersurface $\alpha^T C\alpha = 1$ of constant curvature which is cut out by the integration limits in the space of Feynman parameters. Sides τ_{ij} of this non-Euclidian simplex are equal to the angles between the mass vectors of the basic simplex (see figure 2)

$$C_{ij} = \cos\tau_{ij}. \tag{12}$$

Equivalently, this content is equal to the N-dimensional solid angle at the mass meeting point of the basic simplex.

Then the integral for the case $D = N$ acquires the following form:

$$J(N, p_1, \ldots p_N) = i^{1-2N} \frac{\pi^{N/2}\Gamma(N/2)\Omega^{(N)}}{N!V^{(N)}}; \tag{13}$$

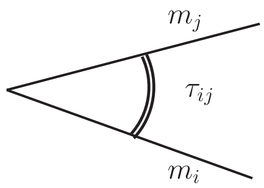

Figure 2. Definition of τ_{ij} in terms of the basic simplex.

hence the calculation of the Feynman integral is nothing but the calculation of the volume in a proper space.

The case $N \neq D$ can be treated similarly with some modifications [20].

Let us turn now to the case of interest that is N-leg MHV amplitudes in four dimensions. The crucial point is that the one-loop MHV amplitudes can be presented as the sum of the two-mass easy box diagrams. These diagrams are IR divergent, that is, it is useful to start with the box diagram with all off-shell particles.

For a box corresponding to $N = 4$,

$$J(4, p_1, p_2, p_3, p_4) = \frac{2\mathrm{i}\pi^2\Omega^{(4)}}{m_1m_2m_3m_4\sqrt{\det C}}. \tag{14}$$

In our formulae we take the mass of the propagators equal to zero. We obtain

$$\left(m_i^2m_2^2m_3^2m_4^2\det C\right)_{m_i\to 0} = \tfrac{1}{16}\lambda\left(k_{12}^2k_{34}^2, k_{13}^2k_{24}^2, k_{14}^2k_{23}^2\right), \tag{15}$$

where the Källen function $\lambda(x, y, z)$ is defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \tag{16}$$

and $\sqrt{-\lambda}$ is just the area of the triangle with sides $\sqrt{k_{12}^2k_{34}^2}, \sqrt{k_{23}^2k_{24}^2}, \sqrt{k_{31}^2k_{23}^2}$.

In the limit of massless propagators $C_{ij} \to \infty$ and should be considered as hyperbolic cosines. Taking them to infinity corresponds to the vertices of the non-Euclidian simplex being located at infinity. Thus, we need to calculate the volume of the ideal hyperbolic tetrahedron. This can be represented through its dihedral angles ψ_{ij}

$$2\mathrm{i}\Omega^{(4)} = \mathrm{Cl}_2(2\psi_{12}) + \mathrm{Cl}_2(2\psi_{13}) + \mathrm{Cl}_2(2\psi_{23}) \tag{17}$$

which are defined via the kinematical invariants

$$\cos\psi_{12} = \frac{k_{13}^2k_{24}^2 + k_{14}^2k_{23}^2 - k_{12}^2k_{34}^2}{2\sqrt{k_{13}^2k_{23}^2k_{14}^2k_{43}^2}} \tag{18}$$

$$\cos\psi_{13} = \frac{k_{14}^2k_{23}^2 + k_{12}^2k_{43}^2 - k_{13}^2k_{24}^2}{2\sqrt{k_{14}^2k_{23}^2k_{12}^2k_{43}^2}} \tag{19}$$

$$\cos\psi_{14} = \frac{k_{12}^2k_{34}^2 + k_{13}^2k_{24}^2 - k_{14}^2k_{32}^2}{2\sqrt{k_{13}^2k_{24}^2k_{12}^2k_{43}^2}} \tag{20}$$

and $\psi_{12} = \psi_{34}, \psi_{13} = \psi_{24}, \psi_{14} = \psi_{32}$.

The functions involved are defined as

$$\mathrm{Cl}_2(x) = \mathrm{Im}[\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}x})] = -\int_0^x \mathrm{d}y\, \ln|2\sin y/2|. \tag{21}$$

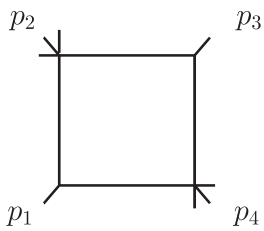

Figure 3. Two-mass easy box diagram.

In the case of the two-mass easy box diagram defining the one-loop MHV amplitude the additional simplification of the kinematical invariants occurs since two external particles are on the mass shell. In this case the arguments of the Li_2 function degenerate to the conformal ratios of four points. The geometrical picture behind the divergent part of the diagram will be discussed later.

Note that the massless 4-point box solution coincides with the 3-point result which has been known for some time [22]. However the geometrical object responsible for the 3-point function is just the triangle. The solution for the generic 3-point function is expressed in terms of the angles of the basic triangle only [20].

The appearance of the hyperbolic volume implies that the topological string approach or CS with the $SL(2, C)$ group are relevant [40]. Indeed we can consider the ideal tetrahedron as the knot complement and calculate it via the Chern–Simons theory action with the complex group. It turns out that the choice of the particular values of the kinematical invariants corresponds to the choice of the particular knot [27].

4. Derivation of the Wilson polygon–MHV amplitude duality at one loop

In this section we shall derive the duality at the one-loop lever via a two-step procedure. First, we describe the change of variables in the space of Feynman parameters which brings the two-mass easy box diagrams into the form of the Wilson polygon in a different dimension. Then we make use of the relation between the Feynman diagrams in $D = 6$ and $D = 4$.

Let us start with the definition of the general box in $D_{\mathrm{IR}} = d_{\mathrm{IR}} - 2\epsilon_{\mathrm{IR}}$ dimensions and use notations from [16]:

$$I(p_i, D_{\mathrm{IR}}, \mu_{\mathrm{IR}}) = -\mathrm{i}\pi^{-\frac{D_{\mathrm{IR}}}{2}} \left(\mu_{\mathrm{IR}}^2\right)^{\epsilon_{\mathrm{IR}}} \int d^{D_{\mathrm{IR}}} l \frac{1}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4)^2} \qquad (22)$$
$$p_i^2 = m_i^2.$$

One can introduce the Feynman parameters and take the integral over l which amounts to

$$I(p_i, D_{\mathrm{IR}}, \mu_{\mathrm{IR}}) = \left(\mu_{\mathrm{IR}}^2\right)^{\epsilon_{\mathrm{IR}}} \Gamma\left(4 - \frac{D_{\mathrm{IR}}}{2}\right) \int \prod \mathrm{d}x_i \frac{\delta(1 - x_1 - x_2 - x_3 - x_4)}{(-\Delta)^{4-\frac{D_{\mathrm{IR}}}{2}}} \qquad (23)$$
$$\Delta = s x_1 x_3 + u x_2 x_4 + m_1^2 x_1 x_2 + m_2^2 x_2 x_3 + m_3^2 x_3 x_4 + m_4^2 x_4 x_1,$$

where $s = (p_1 + p_2)^2$ and $u = (p_2 + p_3)^2$. Let us focus on the two-mass easy box diagram when $m_1^2 = m_3^2 = 0$ (see figure 3) and therefore

$$\Delta_{2me} = s x_1 x_3 + u x_2 x_4 + m_2^2 x_2 x_3 + m_4^2 x_4 x_1.$$

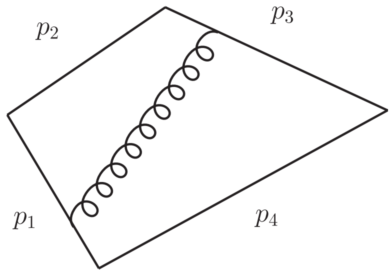

Figure 4. Wilson diagram dual to the finite part of the two-mass easy box.

Upon the following change of variables

$$\begin{aligned} x_1 &= \sigma_1\tau_1 \\ x_2 &= \sigma_1(1-\tau_1) \\ x_3 &= \sigma_2(1-\tau_2) \\ x_4 &= \sigma_2\tau_2 \\ \left|\frac{\partial(x_i)}{\partial(\sigma_i,\tau_i)}\right| &= \sigma_1\sigma_2, \end{aligned} \tag{24}$$

the integration over σ_i factorizes and one obtains

$$I^{2me}(p_i, D_{\mathrm{IR}}, \mu_{\mathrm{IR}}) = \left(\mu_{\mathrm{IR}}^2\right)^{\epsilon_{\mathrm{IR}}}\Gamma\left(4-\frac{D_{\mathrm{IR}}}{2}\right)\int \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\sigma_1^{\frac{D_{\mathrm{IR}}}{2}-3}\sigma_2^{\frac{D_{\mathrm{IR}}}{2}-3}\delta(1-\sigma_1-\sigma_2)$$
$$\times\int_0^1 \mathrm{d}\tau_1\,\mathrm{d}\tau_2\frac{1}{\left(-\left[\left(m_2^2+m_4^2-s-u\right)\tau_1\tau_2+\left(s-m_2^2\right)\tau_1+\left(u-m_2^2\right)\tau_2+m_2^2\right]\right)^{4-\frac{D_{\mathrm{IR}}}{2}}}. \tag{25}$$

In this expression one can observe much similarity with the Wilson loop diagram depicted in figure 4. Indeed, we will show further that proper identification of parameters allows us to connect it with the Wilson loop diagram explicitly.

It is important that the special combinations of Feynman parameters play the role of parametrization of the point in the Wilson polygon which emerges in the one-loop calculation

$$W(\mathcal{C}_n) = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left[\mathrm{i}g\oint \mathrm{d}\tau\,\dot{x}^{\mu}(\tau)A_{\mu}(x(\tau))\right]. \tag{26}$$

We assume that $D_{\mathrm{UV}} = d_{\mathrm{UV}} - 2\epsilon_{\mathrm{UV}}$, p_1 and p_3 are light-like, $x = p_1(1-\tau_1)$, $y = p_1 + p_2 + p_3\tau_2$ and the standard propagator in the Feynman gauge

$$G^F_{\mu\nu}(x-y) = -\eta_{\mu\nu}\frac{\left(\pi\mu_{\mathrm{UV}}^2\right)^{\epsilon_{\mathrm{UV}}}}{4\pi^2}\frac{\Gamma\left(\frac{D_{\mathrm{UV}}}{2}-1\right)}{(-(x-y)^2+i\epsilon)^{\frac{D_{\mathrm{UV}}}{2}-1}}. \tag{27}$$

Ignoring the trivial factor $\frac{g^2C_F}{16\pi^2}$ we obtain the following expression for the diagram:

$$I^W_{ij}(p_i, D_{\mathrm{UV}}, \mu_{\mathrm{UV}}) = \Gamma\left(\frac{D_{\mathrm{UV}}}{2}-1\right)\left(\pi\mu_{\mathrm{UV}}^2\right)^{\epsilon_{\mathrm{UV}}}$$
$$\times\int_0^1 \mathrm{d}\tau_i\,\mathrm{d}\tau_j\frac{m_2^2+m_4^2-s-u}{\left(-\left[\left(m_2^2+m_4^2-s-u\right)\tau_1\tau_2+\tau_1\left(s-m_2^2\right)+\tau_2\left(u-m_2^2\right)+m_2^2\right]\right)^{\frac{D_{\mathrm{UV}}}{2}-1}} \tag{28}$$

and from (25) and (28) we can identify the parameters to enable us to match two expressions. Namely substituting $\frac{D_{\rm UV}}{2} - 1 = 4 - \frac{D_{\rm IR}}{2}$ we obtain

$$\begin{aligned} &d_{\rm UV} + d_{\rm IR} = 10 \\ &\epsilon_{\rm IR} = -\epsilon_{\rm UV} \\ &\left(\mu_{\rm UV}^2 \pi\right)^{\epsilon_{\rm UV}} = \left(\mu_{\rm IR}^2\right)^{\epsilon_{\rm IR}} \end{aligned} \tag{29}$$

and the exact correspondence reads as follows:

$$\begin{aligned} I^{2me}(p_i, D_{\rm IR}, \mu_{\rm IR}) &= \frac{1}{m_2^2 + m_4^2 - s - u} \int_0^1 \mathrm{d}\sigma \sigma^{2-\frac{D_{\rm UV}}{2}} (1-\sigma)^{2-\frac{D_{\rm UV}}{2}} I_{ij}^W(p_i, D_{\rm UV}, \mu_{\rm UV}) \\ &= \frac{1}{m_2^2 + m_4^2 - s - u} \frac{\Gamma\left(3 - \frac{D_{\rm UV}}{2}\right)^2}{\Gamma(6 - D_{\rm UV})} I_{ij}^W(p_i, D_{\rm UV}, \mu_{\rm UV}). \end{aligned}$$

Note that it is possible to represent the expression for the Wilson polygon in a form which involves integrating over the reparametrization of the boundary contour:

$$I^{2me}(p_i, D_{\rm IR}, \mu_{\rm IR}) = \frac{1}{m_2^2 + m_4^2 - s - u} \int_0^1 \mathrm{d}\sigma I^W(\mathcal{C}(\sigma), p_i, D_{\rm UV}, \mu_{\rm UV})$$

$$\mathcal{C}(\sigma) : p_i \to \sqrt{\sigma(1-\sigma)}\, p_i.$$

This form of the solution was suggested for the strong coupling [32] when the integration over the reparametrizations of the boundary of the Wilson loop is necessary to restore the conformal invariance of the solution.

Suppose we are interested in the $D_{\rm UV} = 4 - 2\epsilon$ Wilson loop diagram. Then, using the known connection between the Wilson diagram and the finite part of the box, we obtain

$$\begin{aligned} I_{ij}^W(p_i, 4 - 2\epsilon) &= \left(m_2^2 + m_4^2 - s - u\right) \frac{\Gamma(2 + 2\epsilon)}{\Gamma^2(1+\epsilon)} I^{2me}(p_i, 6 + 2\varepsilon) \\ &= \mathrm{Fin}\left[\frac{\Gamma(1+2\epsilon)}{\Gamma^2(1+\epsilon)} I^{2me}(p_i, 4 + 2\epsilon) \frac{1}{2}\left(m_2^2 m_4^2 - su\right)\right] \end{aligned} \tag{30}$$

and therefore the following relation provides the desired duality

$$I^{2me}(p_i, 6 + 2\epsilon) = \mathrm{Fin}\left[\frac{I^{2me}(p_i, 4 + 2\epsilon)}{1 + 2\epsilon} \frac{\left(su - m_2^2 m_4^2\right)}{2\left(s + u - m_2^2 - m_4^2\right)}\right]. \tag{31}$$

Such a connection between D- and $(D-2)$-dimensional scalar loop integrals indeed exists [38]. Here we are interested in the case of $D = 6$ two-mass easy boxes and their connection with the $D = 4$ ones [15]. The formula reads as follows (see appendix A):

$$I^{2me}(6 + 2\epsilon) = \frac{1}{(1+2\epsilon) z_0}\left(I^{2me}(4 + 2\epsilon) - \sum_{i=1}^{4} z_i I^{2me}(4 + 2\epsilon; 1 - \delta_{ki})\right),$$

where

$$
\begin{aligned}
z_0 &= \sum_{i=1}^{4} z_i = 2\frac{s+u-m_2^2-m_4^2}{su-m_2^2m_4^2}\\
z_1 &= \frac{u-m_2^2}{su-m_2^2m_4^2}\\
z_2 &= \frac{s-m_4^2}{su-m_2^2m_4^2}\\
z_3 &= \frac{u-m_4^2}{su-m_2^2m_4^2}\\
z_4 &= \frac{s-m_2^2}{su-m_2^2m_4^2}.
\end{aligned}
$$

As can be easily seen, the $\sum_{i=1}^{4} z_i I^4(4+2\epsilon; 1-\delta_{ki})$ does precisely the job of taking the finite part.

As we know from the calculation of the one-loop NMHV amplitudes [28], new ingredients emerge, namely two-mass hard and three-mass boxes. Thus if one wants to extend the duality between the Wilson loop and amplitudes to the NMHV case one should be able to get these ingredients from the Wilson loop language.

In the case of the two-mass easy box the structure of the function in the space of Feynman parameters space allows us to use the change of variables to get the Wilson loop diagram multiplied by the simple numerical integral. We can interpret this as an integral over the reparametrizations of the contours. One can try to use the same approach of splitting the Feynman parameters into two pairs: one pair parameterizes the contour while the second yields the standard parametrization of points where the gluon propagator is attached.

For two-mass hard, three- and four-mass boxes factorization fails and therefore the simple geometrical interpretation does not work. Namely, if we make all legs massive in the Feynman box and consider the corresponding Wilson contour, the integrands in the amplitude and the Wilson loop look as follows:

$$
\begin{aligned}
\Delta_W &= -\left(s+u-m_2^2-m_4^2\right)\tau_1\tau_2 + \left(u-m_2^2\right)\tau_1 + \left(s-m_2^2\right)\tau_2 + m_2^2\\
&\qquad - m_1^2\tau_1\left(1-\tau_1\right) - m_3^2\tau_2(1-\tau_2)
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\Delta_A &= \sigma_1\sigma_2\left[-\left(s+u-m_2^2-m_4^2\right)\tau_1\tau_2 + \left(u-m_2^2\right)\tau_1 + \left(s-m_2^2\right)\tau_2 + m_2^2\right]\\
&\quad + m_1^2\sigma_1^2\tau_1(1-\tau_1) + m_3^2\sigma_2^2\tau_2(1-\tau_2)\\
&= \sigma_1\sigma_2\Delta_W + m_1^2\sigma_1\tau_1(1-\tau_1) + m_3^2\sigma_2\tau_2(1-\tau_2).
\end{aligned}
\tag{33}
$$

We have not found a simple geometrical interpretation of transformation from Δ_W to Δ_A in terms of the reparametrizations of the Wilson contour and we cannot naturally connect two-mass hard and harder boxes diagrams with Wilson diagrams for the corresponding contours. That is, if a connection between NMHV amplitudes and Wilson polygon-like objects exists, which is expected according to the T-dual picture of $\mathrm{AdS}_5 \times S_5$ superstring [12], then it seems to be more complicated.

5. 3-point function–Wilson triangle duality

In this section we consider the example of the similar duality for the 3-point function and it will be clear how the generalization of the duality for the 'two-mass hard' diagram involves

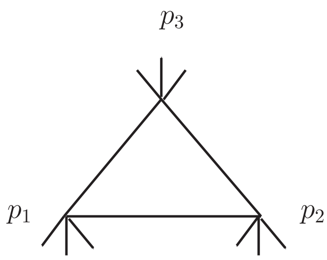

Figure 5. Scalar triangle diagram.

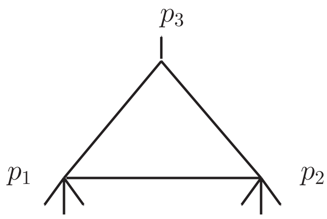

Figure 6. Two-mass triangle diagram.

the particular vertex operator. To start with let us mention also the interesting relation between the one-loop 3-point amplitude and the two-loop vacuum energy in the scalar theory. Namely, if one considers the 3-point function $I\left(p_1^2, p_2^2, p_3^2\right)$ with the external virtualities p_1^2, p_2^2, p_3^2 and the two-loop vacuum energy $J\left(m_1^2, m_2^2, m_3^2\right)$ with the masses of the three internal propagators m_1^2, m_2^2, m_3^2 then the following relation holds [21]:

$$I\left(D = 4 - 2\epsilon, p_1^2, p_2^2, p_3^2\right) = J\left(4 + 2\epsilon, m_1^2, m_2^2, m_3^2\right). \tag{34}$$

That is, the duality discussed below can be applied both for the one-loop amplitude and the two-loop vacuum energy.

Consider the most general triangle in the massless scalar theory (see figure 5). In the Feynman parametrization it is equal to

$$p_1 + p_2 + p_3 = 0$$

$$I_\triangle(p_i, D_{\mathrm{IR}}, \mu_{\mathrm{IR}}) = -\left(\mu_{\mathrm{IR}}^2\right)^{\epsilon_{\mathrm{IR}}} \Gamma\left(3 - \frac{D_{\mathrm{IR}}}{2}\right) \int \prod \mathrm{d}x_i \frac{\delta(1 - x_1 - x_2 - x_3)}{(-\Delta)^{3 - \frac{D_{\mathrm{IR}}}{2}}} \tag{35}$$

$$\Delta = m_3^2 x_1 x_2 + m_2^2 x_1 x_3 + m_1^2 x_2 x_3$$

and assuming $p_3^2 = 0$ (see figure 6) we have

$$\Delta = m_2^2 x_1 x_3 + m_1^2 x_2 x_3.$$

Let us make the following change of variables:

$$x_1 = \sigma(1 - \tau) \qquad x_2 = \sigma\tau \tag{36}$$

which amounts to

$$I_\triangle(p_i, D_{\mathrm{IR}}, \mu_{\mathrm{IR}}) = \left(\mu_{\mathrm{IR}}^2\right)^{\epsilon_{\mathrm{IR}}} \Gamma\left(3 - \frac{D_{\mathrm{IR}}}{2}\right) \int \mathrm{d}\sigma\, \mathrm{d}x_3 \sigma \frac{\delta(1 - \sigma - x_3)}{(\sigma x_3)^{3 - \frac{D_{\mathrm{IR}}}{2}}}$$

$$\times \int_0^1 \mathrm{d}\tau \frac{1}{\left(m_2^2(1 - \tau) + m_1^2\tau\right)^{3 - \frac{D_{\mathrm{IR}}}{2}}}. \tag{37}$$

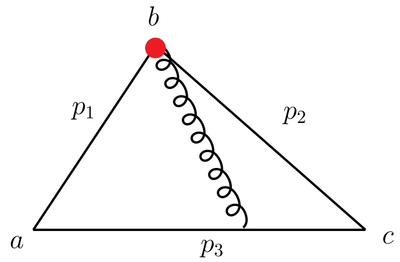

Figure 7. Wilson diagram dual to the two-mass triangle diagram.

In Wilson-dual language we can interpret it in the following way (see figure 7):

$$m_2^2(1-\tau)+m_1^2\tau=(p_2+p_3\tau)^2 \tag{38}$$

and the identification of parameters reads as follows:

$$\begin{aligned} &d_{\mathrm{UV}}+d_{\mathrm{IR}}=8\\ &\epsilon_{\mathrm{IR}}=-\epsilon_{\mathrm{UV}}\\ &\left(\mu_{\mathrm{UV}}^2\pi\right)^{\epsilon_{\mathrm{UV}}}=\left(\mu_{\mathrm{IR}}^2\right)^{\epsilon_{\mathrm{IR}}}; \end{aligned} \tag{39}$$

therefore this diagram can be understood assuming the presence of the vertex operator

$$\left\langle Tr\mathcal{P}q^\mu A_\mu(x_b)\exp\left[ig\oint_C \mathrm{d}\tau\dot{x}^\mu(\tau)A_\mu(x(\tau))\right]\right\rangle, \tag{40}$$

where q^μ can be chosen to be the arbitrary vector which is not orthogonal to p_3 in the Minkowski sense, $(p_3q)\neq 0$. This q^μ can naturally be identified with the polarization vector of the correspondent external gluon. Hence we have an example of the possible extension of the Wilson dual side, when it becomes sensitive to polarizations of external gluons. This example provides some intuition for the possible generalization of the duality to less symmetric theories or NMHV amplitudes. Nevertheless, the problem of interpreting the three-mass triangle and all boxes harder than the two-mass easy one in terms of Wilson loop diagrams is still open.

6. Analytical structure of the light-like Wilson loop

6.1. General comments

In this section we discuss the analytical structure of the light-like Wilson loop. If the correspondence between MHV amplitudes and Wilson loops is true at any order of perturbation theory, obviously their analytical structure, namely the location of singularities, branches and discontinuities in the space of kinematic moduli, should match each other. Thus there emerge two interesting problems in themselves: analytical structure of perturbative light-like Wilson loop and a similar question concerning the areas in AdS_5 bounded by the light-like contour.

Here we begin the analysis of the analytical structure of the perturbative Wilson loop. First, we can do it using its connection with scattering amplitudes. The fact of unitarity of QFT leads to the optical theorem and allows one to take different branch cuts and develop the generalized unitarity method to simplify loop computations. Using the correspondence between Wilson loops and amplitudes we can reformulate the optical theorem at one loop in terms of Wilson loops.

Second, one can analyze the analytical structure of every Wilson diagram on its own. The systematic method of clarifying the structure of singularities of Feynman amplitudes was developed a long time ago in the theory of an analytic S-matrix. It can obviously be applied to the Wilson loop diagrams. At the one-loop level, using the results of the previous section, we can apply the Cutkosky rules to $10 - D_W$ boxes which are dual to the Wilson diagrams to get the result for the given diagram, while at higher orders additional arguments are required.

6.2. Landau singularities for the Wilson loop

The approach considered here is parallel to that of [35], where an excellent introduction to the problem can be found. If we deal with scalar massless theory Feynman integrals in D dimensions, then we have, for any diagram [36],

$$I \simeq \int_0^1 \prod \mathrm{d}\alpha_i \delta\left(1 - \sum_i \alpha_i\right) \frac{\mathcal{U}^{N-(L+1)\frac{D}{2}}}{(-\mathcal{V})^{N-L\frac{D}{2}}}. \tag{41}$$

Here L is the number of loops; N is the number of propagators; α_i corresponds to the ith propagator in the diagram of the form $\frac{1}{q_i^2}$,

$\mathcal{U} = \sum_{T\in T_1} \prod_{i\in\bar{T}} \alpha_i$ is the sum over the so-called 1-trees, degree L in α,

$\mathcal{V} = \sum_{T\in T_2} \prod_{i\in\bar{T}} \alpha_i (Q_T)^2$ the sum over the so-called 2-trees, degree$(L+1)$ in α,

and I can be considered as the function of complex kinematical parameters. A natural question arises: where are its singularities located in the space of parameters? The answer to this question is given by the Landau equations which can be written in the following form:

$$\begin{cases} \dfrac{\partial\mathcal{V}}{\partial\alpha_i} = 0 \wedge \alpha_i = 0 \\ \mathcal{V} = 0, \end{cases}$$

and the same analysis can be applied for any particular Wilson loop. If one considers a family of more simple diagrams where every propagator has one leg lying on the boundary, then the consideration is in full analogy with the case of amplitudes. Namely the diagram has the following structure

$$W \simeq \int_0^1 \prod_{i=1}^{2N} \mathrm{d}\tau_i \theta_{\mathrm{Path}}\left(x\left(\tau_{\sigma_1}\right) > x\left(\tau_{\sigma_2}\right) > \cdots > x\left(\tau_{\sigma_{2N}}\right)\right) \prod_{k=1}^{V_3} \hat{L}_k \frac{d^D z_1 d^D z_2 \ldots d^D z_V}{\prod_{k=1}^N \left(-x_k^2\right)^{\frac{D}{2}-1}}. \tag{42}$$

Here $\hat{L}_k$ is the differential operator independent of z_i, which comes from three-gluon vertices [39]; V_3, the number of three-gluon vertices

$$\begin{aligned} A^{\mu_1} A^{\mu_2} A^{\mu_3} \int d^D z_k Tr[\partial_\mu (A_\nu [A^\mu, A^\nu])(z_k) \sim \big[\eta^{\mu_1\mu_2}\left(\partial_1^{\mu_3} - \partial_2^{\mu_3}\right) + \eta^{\mu_2\mu_3}\left(\partial_1^{\mu_1} - \partial_2^{\mu_1}\right) \\ + \eta^{\mu_1\mu_3}\left(\partial_1^{\mu_2} - \partial_2^{\mu_2}\right)\big] G(x_1, x_2, x_3) \hat{L}_k^{\mu_1\mu_2\mu_3} G(x_1, x_2, x_3). \end{aligned} \tag{43}$$

Then if the points (x_1, x_2, x_3) lie on the edges (y_1, y_2, y_3),

$$\hat{L}_k = \dot{y}_{1\mu_1} \dot{y}_{2\mu_2} \dot{y}_{3\mu_3} \hat{L}_k^{\mu_1\mu_2\mu_3}. \tag{44}$$

For any ordering, there exists a change of variables of integration with the Jacobian J, independent of the kinematical variables, that makes the simple integration interval:

$$\int_0^1 \prod_{i=1}^{2N} \mathrm{d}\tau_i \theta_{\mathrm{Path}}(\tau) \to \int_0^1 \prod_{i=1}^{2N} \mathrm{d}\tilde{\tau}_i J(\tilde{\tau}). \tag{45}$$

If we have the following ordering along one of the edges $\int_0^1 \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \ldots \int_0^{\tau_2} \mathrm{d}\tau_1$, then we can choose

$$\begin{aligned}
&\tau_n = \tilde{\tau}_n \\
&\tau_{n-1} = \tau_n \tilde{\tau}_{n-1} = \tilde{\tau}_n \tilde{\tau}_{n-1} \\
&\ldots \\
&\tau_1 = \tau_2 \tilde{\tau}_1 = \tilde{\tau}_n \tilde{\tau}_{n-1} \ldots \tilde{\tau}_1 \\
&J(\tilde{\tau}) = \prod_{j=1}^{n} \tilde{\tau}_j^{\,j-1}.
\end{aligned} \tag{46}$$

If the vertices are absent then the Landau equations take the form

$$\begin{cases}
\frac{\partial\left(\sum \alpha_k x_k^2\right)}{\partial \alpha_i} = 0 \wedge \alpha_i = 0 \\
\frac{\partial\left(\sum \alpha_k x_k^2\right)}{\partial \tilde{\tau}_i} = 0 \wedge \tilde{\tau}_i = 0 \wedge \tilde{\tau}_i = 1 \\
\sum \alpha_k x_k^2 = 0.
\end{cases}$$

In the presence of vertices we can introduce the Feynman parameters

$$W \simeq \int_0^1 \prod_{i=1}^{2N} \mathrm{d}\tilde{\tau}_i J(\tilde{\tau}) \prod_{k=1}^{V_3} \hat{L}_k \int_0^1 \prod_{k=1}^{N} \mathrm{d}\alpha_k \alpha_k^{\frac{D}{2}-2} \delta\left(1 - \sum_i \alpha_i\right) \frac{d^D z_1 d^D z_2 \ldots d^D z_V}{\left[-\sum \alpha_k x_k^2\right]^{N(\frac{D}{2}-1)}} \tag{47}$$

and integration over the vertex position can be performed, yielding the solution

$$W \simeq \int_0^1 \prod_{i=1}^{2N} \mathrm{d}\tilde{\tau}_i J(\tilde{\tau}) \prod_{k=1}^{V_3} \hat{L}_k \int_0^1 \prod \mathrm{d}\alpha_k \alpha_k^{\frac{D}{2}-2} \delta\left(1 - \sum_i \alpha_i\right) \frac{\mathcal{U}_W(\alpha_i)}{(-\mathcal{V}_W)^{(N-V)\frac{D}{2}-N}}. \tag{48}$$

Here, V is the number of vertices; N is the number of propagators; α_i corresponds to the ith propagator in the diagram of the form $\frac{1}{(-x_i^2)^{\frac{D}{2}-1}}$; $\mathcal{U}_W$ and $\mathcal{V}_W$ are the result of the integration over the loop momenta.

Finally we obtain the following Landau equations:

$$\begin{cases}
\frac{\partial \mathcal{V}_W}{\partial \alpha_i} = 0 \wedge \alpha_i = 0 \\
\frac{\partial \mathcal{V}_W}{\partial \tilde{\tau}_i} = 0 \wedge \tilde{\tau}_i = 0 \wedge \tilde{\tau}_i = 1 \\
\mathcal{V}_W = 0.
\end{cases}$$

6.3. Imaginary part of the Wilson loop at one loop

In unitary theory one can exploit the unitarity of the S-matrix to obtain the following identity:

$$\begin{aligned}
&S^+ S = 1 \qquad S = 1 + \mathrm{i}T \\
&2\mathrm{Im}(\mathcal{A}(\text{in} \to \text{out})) = -i(\mathcal{A}(\text{in} \to \text{out}) + \mathcal{A}^*(\text{out} \to \text{in})) \\
&\qquad = \sum_{\text{states}} \mathcal{A}^*(\text{out} \to \text{all})\mathcal{A}(\text{in} \to \text{all}).
\end{aligned} \tag{49}$$

We are interested in amplitudes with n outgoing particles. In this case, the RHS sum becomes the sum over the state with integration over a Lorentz invariant phase space corresponding to the final particles. Extending this statement to the diagrammatic level one ends up with Cutkosky rules and a prescription of the cutting propagators:

$$\frac{1}{k^2 + \mathrm{i}\epsilon} \to \theta(k_0)\delta(k^2). \tag{50}$$

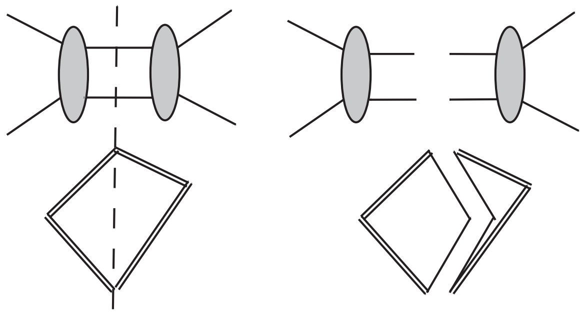

Figure 8. Wilson loop analog of cut rules for scattering amplitudes.

It is well known that to get the imaginary part of the diagram to a given order one should sum over all possible cuts of all diagrams and over all possible intermediate states. Then one should do the integration over LIPS. The Wilson loop detects $\mathcal{N} = 4$ SYM particle content only through correction to the gluon propagator and the vertices. At the one-loop level it is obviously insensitive to particle content. On the amplitude side the cut is in contrast essentially dependent on the particle content and tree-level amplitude even at one loop. According to the strong version of the correspondence which is true at one loop,

$$\mathcal{A}_n^{\mathrm{MHV}} = \mathcal{A}_n^{\mathrm{tree}} W(\mathcal{C}_n) \tag{51}$$

with necessary identification of parameters. Since $\mathcal{A}_n^{\mathrm{tree}}$ is a rational function of kinematical variables it does not contribute to the cut. Hence we can rewrite the optical theorem as

$$\begin{aligned} 2\mathrm{Im}[W(\mathcal{C}_n)] &= 2\mathrm{Im}\left(\frac{\mathcal{A}(\mathrm{in} \to \mathrm{out})}{\mathcal{A}^{\mathrm{tree}}(\mathrm{in} \to \mathrm{out})}\right) \\ &= \sum_{\mathrm{states}} \frac{\mathcal{A}^*(\mathrm{out} \to \mathrm{all})\mathcal{A}(\mathrm{in} \to \mathrm{all})}{\mathcal{A}^{\mathrm{tree}}(\mathrm{in} \to \mathrm{out})}. \end{aligned} \tag{52}$$

At the one-loop level we have sum over products of tree-level amplitudes. Denoting this sum divided by $\mathcal{A}_n^{\mathrm{tree}}$ as V_W (which can be found in appendix B) we have

$$\mathrm{Im}[W(\mathcal{C})] = \int_{\mathcal{C}_L \mathcal{C}_R} V_W(\mathcal{C}_L, \mathcal{C}_R), \tag{53}$$

where integration goes over contours which one could get by breaking the loop into two parts, inserting a special vertex, which one could find from summing over states in $\mathcal{N} = 4$ SYM, and then by integration over contours which are limited by momentum conservation and the light-like condition for every edge (see figures 8 and 9).

On the other hand the problem of finding the imaginary part can be considered at the diagrammatical level where the connection with the box in dual dimension makes it possible to apply Cutkosky rules. Of course, in this method there is no summation over states. It would be nice to understand how the vertices from the dual amplitude picture occur in this scenario. The following picture arises if one considers the quadruple cut. It is interesting to note the role of the coefficient

$$\mathcal{C}_{\mathrm{MHV}}^{2me} = \delta^8\left(\sum_{i=1}^{n} \lambda_i \eta_i\right) \Delta \tag{54}$$

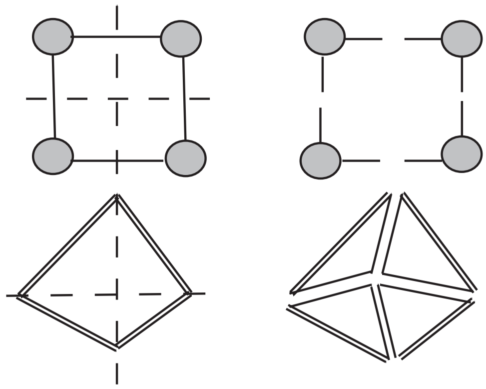

Figure 9. Four-particle cut and its Wilson loop counterpart.

which appears from the quadruple cut of the MHV amplitude and is defined by the structure of tree amplitudes in $\mathcal{N} = 4$ SYM. In the Wilson loop calculation which is blind to trees it appears as we go down from $D = 6$ to $D = 4$ dimensions, namely

$$z_0 \sim \frac{1}{\Delta}. \tag{55}$$

7. On the geometry of UV/IR divergences

Let us discuss the interpretation of the divergent contributions. The IR singularity of the amplitude corresponds to the UV singularity of the cusps, hence the very issue of the proper IR regularization of the amplitude is essentially related to the smoothing of the cusps in the polygon in the momentum space.

Let us make a few comments concerning the proper identification of the cusp anomaly in geometrical terms [25, 26]. Since the amplitude is expressed in terms of the hyperbolic volumes and area in 3D AdS space, it is natural to question what the cusp anomaly corresponds to in the same setting. That is, we can start with a box with all external momenta off-shell which is finite. Then approaching the on-shell limit for two external momenta the geometrical volume and area start diverging which corresponds to the divergence of the Feynman diagram. Nevertheless we expect that the initial geometry is partially seen in the divergent terms.

Recall that $\Gamma_{\text{cusp}}(\theta, \alpha)$ is the cusp anomalous dimension which for the cusp angle θ at one loop behaves as

$$\Gamma_{\text{cusp}}(\theta, \alpha) = \frac{\alpha C_F}{\pi}(\theta \coth\theta - 1) + O(\alpha^2). \tag{56}$$

It turns out [25] that the one-loop expression is nothing but the transition amplitude in AdS_3

$$\Gamma_{\text{cusp}}(\theta) \propto \langle v'|1/\Delta_{S_3}|v\rangle, \tag{57}$$

where two light-like vectors v and v' cross at the angle, and Δ_{S_3} is the corresponding Laplace operator on the $SU(2)$ group manifold. That is, the one-loop anomaly can be attributed to the amplitude along the single edge of the basic simplex upon the analytic continuation [25]. Note that these geodesics connecting two vertices are dressed by the quadratic fluctuations.

Since the quantum geometry of the AdS_3 is governed by the $SL(2, C)$ Chern–Simons theory the corresponding Wilson loop is just the particle moving in this background. It is also possible to make the link with the AdS_2 geometry since the one-loop cusp anomaly can be written as the wave functional in the two-dimensional YM theory on the disc integrated over its area:

$$\Gamma_{\text{cusp}}(\theta) \propto \int \mathrm{d}A(Z(U, A) - Z(U, 0)), \tag{58}$$

where A is the area of the disc, U is the boundary holonomy and $Z(U, A)$ is the partition function of the 2D YM theory in the disc geometry.

Since it is expected that the reparametrization of the boundary enters the solution it is natural to search for the Liouville interpretation of the cusp anomaly. Contrary to the finite contribution where the reparametrization part decouples and does not depend on the kinematical invariants, we expect that the divergent 'Liouville' contribution has nontrivial kinematical dependence.

Possible arguments which deserve more justification are as follows [26]. Consider two-dimensional scalar field theory with the equation of motion

$$\left(\partial_t^2 - \partial_x^2\right)\phi + m^2\phi = 0 \tag{59}$$

whose solution has the following mode expansion

$$\phi(x, t) = \int \frac{\mathrm{d}\beta}{2\pi}(a^*(\beta)\,\mathrm{e}^{-\mathrm{i}m(x\sinh\beta - t\cosh\beta} + a(\beta)\,\mathrm{e}^{\mathrm{i}m(x\sinh\beta - t\cosh\beta}). \tag{60}$$

It is convenient to introduce Rindler coordinates

$$\begin{aligned} &x = r\cosh\theta, && t = r\sinh\theta \\ &-\infty < \theta < +\infty && 0 < r < +\infty \end{aligned} \tag{61}$$

in the space-time region $x > |t| > 0$. Upon the following Laplace transform with respect to the radial coordinate,

$$\lambda_\theta(\alpha) = \int \mathrm{d}r\,\mathrm{e}^{imr\sinh\alpha}\left(-\frac{1}{r}\partial_\theta + \mathrm{i}m\cosh\alpha\right)\phi(r, \theta), \tag{62}$$

the commutation relation for the Laplace transformed field reads as

$$[\lambda(\alpha_1), \lambda(\alpha_2)] = \mathrm{i}\hbar\tanh(\alpha_1 - \alpha_2)/2 \tag{63}$$

and the Hilbert space is spanned by vectors $a(\beta_n)\ldots a(\beta_1)|\text{vac}\rangle$ where the vacuum state is defined as

$$a(\beta)|\text{vac}\rangle = 0 \qquad \langle\text{vac}|a^+(\beta) = 0. \tag{64}$$

One can introduce the two-point function on the 'rapidity plane'

$$F(\alpha_1 - \alpha_2) = \langle\text{vac}|\lambda(\alpha_1)\lambda(\alpha_2)|\text{vac}\rangle \tag{65}$$

and an explicit calculation gives the following result [42]

$$F(\alpha - \mathrm{i}\pi) = -\frac{1}{\pi}\alpha/2\coth(\alpha/2) + \text{singular terms}. \tag{66}$$

Hence the singular terms cancel in the difference $F(\alpha - \mathrm{i}\pi) - F(-\mathrm{i}\pi)$ which coincides with the cusp anomaly in agreement with the interpretation of [25] in the first quantized picture.

The relation with the Liouville model becomes clear upon the proper limiting procedure. To this end we can try to represent the Klein–Gordon equation of motion as the zero curvature condition for $SL(2, R)$ connection. Similarly the equation of motion in the Liouville model

$$\left(\partial_t^2 - \partial_x^2\right)\phi + \frac{m^2}{b}\,\mathrm{e}^{b\phi} = 0 \tag{67}$$

can be considered as a zero curvature condition for the $SL(2, R)$-valued connection A_θ, A_r. It is convenient to introduce the monodromy matrix in the Liouville model

$$\mathbf{T}^\theta(\alpha) \propto \mathrm{e}^{\mathrm{i}mR\sinh\alpha\sigma_3}\mathcal{P}\exp\left(\int \mathrm{d}r\, A_r(r,\theta,\alpha)\right), \tag{68}$$

where R is the cutoff, which defines $\lambda_{\mathrm{Liouv}}(x)$ via the relation

$$\lambda(\alpha) = -\mathrm{i}\ln T_{11}(\alpha), \tag{69}$$

and the latter reduces to the corresponding Klein–Gordon function involved into the cusp anomaly and in the weak coupling limit $b \to 0$ [42]

$$\lambda_{\mathrm{Liouv}}(\alpha) \to \frac{b}{4}\lambda_{\mathrm{KG}}(\alpha). \tag{70}$$

8. Discussion

In this paper we have discussed different aspects of the duality between the calculation of the Wilson polygons and amplitudes in SUSY gauge theories focusing mainly on the one-loop correspondence. It turns out that the duality for the MHV amplitude can explicitly be derived in the one-loop case. The derivation is remarkably simple and involves only the proper change of the variables and the relation between the Feynman integrals in the different space-time dimensions. The Wilson polygon to some extent can be thought of as placed in the space of the Feynman parameters and it is in this space that the change of variables is important. It was shown that the UV behavior of the Wilson polygon precisely maps into the IR behavior of the amplitude which explains the correspondence between the regularizations observed earlier.

The change of variables works well for the MHV amplitude only, which can be expressed in terms of two-mass easy box diagrams and the generalization of the duality for the NMHV cases is required. Note that we have identified the key feature of the MHV kinematics—only in this case the integration over reparametrizations is decoupled which is not true for the other cases. Therefore one could expect for the generic kinematics the emergence of the correlators of the Liouville modes responsible for the reparametrizations of the boundary contour. We consider a similar duality for the 3-point function with one external particle on-shell. It was shown that the duality can be formulated upon the insertion of the particular vertex operator into the Wilson triangle. We consider this example as providing a possible method for the generalization of the duality for the polarization-sensitive case. Let us emphasize that SUSY was not essentially used in our one-loop derivation of the duality. Probably the duality can be similarly developed for the non-SUSY theories as well.

It is worth making a more general comment concerning the relation of our analysis with the moduli space geometry. In the approach of [17, 18], the Schwinger parameters get mapped generically into the radial coordinate in AdS_5 and the moduli space of the complex structures $M_{g,n}$ where n is related to the number of external legs in the amplitude. That is, the Schwinger parametrization is closely related to the B model. On the other hand, in our paper we exploited a picture with the emergent Kähler moduli which happens in the A model. In principle one could imagine that a kind of mirror transform on the level of the Feynman diagrams can be formulated and it would be very interesting to investigate this issue further. Note also that the A model under consideration allows the target space an effective description in terms of the effective noncommutative gauge theory [41]. We hope to discuss the possible relation between the Wilson loops in $D = 6$ we have discussed with the corresponding object in the effective target space $D = 6$ gauge theory elsewhere.

The duality implies that a kind of unitary technique can be developed for the calculation of the Wilson polygon as well. We have formulated the cut procedure for the one-loop Wilson polygon which involves integration along the cut with the particular vertex-like operator. Along this line of reasoning we have also derived the analog of the Landau equations for the singularities for the Wilson polygon in terms of the Feynman parameters. Let us emphasize that the geometry behind the Landau equations has a lot in common with the generic hyperbolic geometry behind the one-loop amplitudes. Actually the generic off-shell box diagram calculates the hyperbolic volume of the simplex defined by the kinematical invariants, that is all divergences emerging upon some external particle that tends to be on-shell are expected to carry some geometrical information about the initial hyperbolic geometry. We have shown that at the one-loop level this indeed happens.

In this paper we have discussed the one-loop case only, hence it would be very interesting to extend this analysis to higher loops. The approach to the all-loop answer based on the quantum geometry of the momentum space suggested in [24] could be useful. Another promising development concerns the relation with the geometry of the knots which emerges because of the relation with the volumes of the hyperbolic spaces identified with the knot complements.

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Appendix A. Connection of scalar integrals in different dimensions

Here we briefly explain the connection between the scalar integrals in different dimensions [15]. If we have the following scalar integral (for kinematical notations see figure A1)

$$I^N(D;\nu_k) \equiv -\mathrm{i}\pi^{-\frac{D}{2}}(\mu^2)^{\epsilon}\int d^D l \frac{1}{A_1^{\nu_1}A_2^{\nu_2}\dots A_N^{\nu_N}} \tag{A.1}$$

then it can be shown that

$$I^N(D-2;\nu_k) = \sum_{i=1}^{N} z_i I^N(D-2;\nu_k-\delta_{ki}) + \left(D-1-\sum_{j=1}^{N}\nu_j\right) z_0 I^N(D;\nu_k), \tag{A.2}$$

where

$$\sum_{i=1}^{N}(r_i-r_j)^2 z_i = 1 \qquad z_0 = \sum_{i=1}^{N} z_i. \tag{A.3}$$

In the main body of the text we choose $D = 6+2\epsilon$, $N = 4$, $\nu_i = 1$.

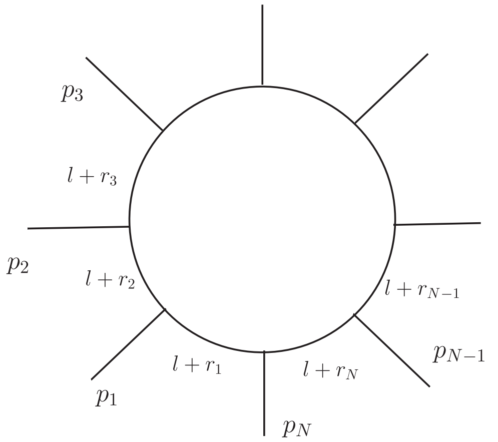

Figure A1. General one-loop scalar diagram.

Appendix B. Sum over states in terms of dual superconformal invariants

It is convenient to use the $\mathcal{N}=4$ on-shell formulation of $\mathcal{N}=4$ SYM, in which all states are encoded in one super-wavefunction

$$\Phi(p,\eta) = G^{+}(p) + \eta^{A}\Gamma_{A}(p) + \frac{1}{2}\eta^{A}\eta^{B}S_{AB}(p) + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\widetilde{\Gamma}^{D}(p) + \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}G^{-}(p). \tag{B.1}$$

We are interested in the cuts of superamplitudes for n particles

$$\mathcal{A}_n(\lambda,\tilde{\lambda},\eta) = \mathcal{A}_n(\Phi_1,\ldots,\Phi_n). \tag{B.2}$$

Using the superamplitude formalism one can easily obtain particular configuration of states using known projectors. As usual, we use the two-component spinor formalism, where $p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}$ and $\langle i|j\rangle = \lambda_i^{\alpha}\lambda_{j\alpha}$.

The N^k MHV amplitude can be presented in terms of nested sums, which are quite cumbersome expressions for MHV and NMHV cases, and are pretty simple

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^{(4)}\big(\sum_{i=1}^{n}\lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}\big)\delta^{(8)}\big(\sum_{i=1}^{n}\lambda_i^{\alpha}\eta_i^{A}\big)}{\langle 1|2\rangle\ldots\langle n-1|n\rangle\langle n|1\rangle}, \tag{B.3}$$

where the second (Grassmann) delta-function makes the supersymmetry manifest

$$\mathcal{A}_n^{\mathrm{NMHV}} = \frac{\delta^{(4)}\big(\sum_{i=1}^{n}\lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}\big)\delta^{(8)}\big(\sum_{i=1}^{n}\lambda_i^{\alpha}\eta_i^{A}\big)}{\langle 1|2\rangle\ldots\langle n-1|n\rangle\langle n|1\rangle}\sum_{(i,j)} R_{k;ij} \tag{B.4}$$

and all indices are understood in the following way $i+n\equiv i$. Then $k+2\leqslant i<j\leqslant n+k-1$ and $j-i\geqslant 2$. $R_{k,ij}$ are dual conformal invariants which are given by the following expressions

$$R_{k;ij} = \frac{\langle i|i-1\rangle\langle j|j-1\rangle\delta^{(4)}(\Xi_{k;ij})}{x_{ij}^2\langle k|x_{ki}x_{ij}|j\rangle\langle k|x_{ki}x_{ij}|j-1\rangle\langle k|x_{kj}x_{ji}|i\rangle\langle k|x_{kj}x_{ji}|i-1\rangle}. \tag{B.5}$$

Here the $\Xi_{k;ij}$ is

$$\Xi_{k;ij} = \langle k|x_{ki}x_{ij}|\theta_{jk}\rangle + \langle k|x_{kj}x_{ji}|\theta_{ik}\rangle; \tag{B.6}$$

thus in this language R depends on $n-2$ momenta.

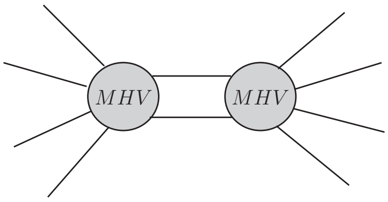

Figure B1. Example of the MHV–MHV cut.

For supermomentum delta-functions we will widely use the following identity [31] which can be easily proved:

$$\delta^{(8)}(I)\delta^{(8)}(J) = \delta^{(8)}(I+J)\delta^{(8)}(J), \tag{B.7}$$

the summation over states is equivalent to the integration over $\int d^4\eta_{\rm cut}$. For particular cuts, it was done in terms of the MHV vertex expansion in [30]. Here we present some results in simple form using the language of dual conformal invariants R.

Thus, the summation over states in the L-loop cut is obtained by

$$\mathcal{A}^{\rm sum}_{\rm cut} = \int d^4\eta_{{\rm cut},1} d^4\eta_{{\rm cut},2} d^4 \ldots \eta_{{\rm cut},L} d^4\eta_{{\rm cut},L+1} \mathcal{A}^{\rm tree}_{\rm left} * \mathcal{A}^{\rm tree}_{\rm right}. \tag{B.8}$$

For N^{L-1} MHV $\times$ MHV cuts using (B.7) we can interpret the summation over states as the action of projection operators.

The result for the N^{L-1} MHV $\times$ MHV cut thus reads as

$$\begin{aligned}\mathcal{A}^{\rm sum}_{\rm cut} = \frac{\delta^{(8)}(\rm ext)}{\langle l_L | l_{L+1}\rangle^4} & A^{\rm gluons,tree}_{N^{L-1}{\rm MHVsplit}}(++\ldots+_{\rm ext}; --\ldots-_{\rm loop}) \\ & \times A^{\rm gluons,tree}_{\rm MHV,right}(--+\ldots+_{\rm loop}; +\ldots+_{\rm ext}).\end{aligned} \tag{B.9}$$

If we cut the MHV diagram at one loop (suppose loop momenta are l_1 and l_2) then for the state sum we have (see figure B1)

$$\mathcal{A}^{\rm sum}_{\rm cut} = \frac{\delta^{(8)}(\rm ext)}{\langle l_1 | l_2\rangle^4} A^{\rm gluons,tree}_{\rm MHV}(++\ldots+_{\rm ext}; --_{\rm loop}) A^{\rm gluons,tree}_{\rm MHV,right}(--_{\rm loop}; +\ldots+_{\rm ext}) \tag{B.10}$$

which agrees with formulae obtained in the literature.

For our purposes we need to divide it on the tree-level amplitude which we have cut. That would be the operator which glues together the Wilson loops:

$$\begin{aligned}V_W &= \frac{\prod_{\rm ext}\langle i|i+1\rangle}{\langle l_1|l_2\rangle^4} A^{\rm gluons,tree}_{\rm MHV}(++\ldots+_{\rm ext}; --_{\rm loop}) A^{\rm gluons,tree}_{\rm MHV,right}(--_{\rm loop}; +\ldots+_{\rm ext}) \\ &= \frac{\langle i|i-1\rangle\langle j|j-1\rangle}{\langle l_1|j\rangle\langle l_1|j-1\rangle\langle l_2|i\rangle\langle l_2|i-1\rangle}\langle l_1|l_2\rangle^2.\end{aligned} \tag{B.11}$$

Another case of special interest for us is the anti-MHV $\times$ MHV cut when we obtain

$$\mathcal{A}^{\rm sum}_{\rm cut} = \frac{\delta^{(8)}(\rm ext)}{\langle l_L | l_{L+1}\rangle^4} A^{\rm gluons,tree}_{\rm MHV}(++_{\rm ext}; --\ldots-_{\rm loop}) A^{\rm gluons,tree}_{\rm MHV,right}(--+\cdots+_{\rm loop}; +\cdots+_{\rm ext}). \tag{B.12}$$

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